Rational Functions: Partial Fraction Decomposition

The Singular Part at a Finite Pole, and Assembling the Decomposition

Lecture Notes

September 29, 2026

1 Partial Fraction Decomposition (continued)

Singular Part at a Finite Pole

Let z_0 be a (finite) pole of R. Consider

$$R_1(\zeta) := R\Big(z_0 + \frac{1}{\zeta}\Big),$$

a rational function of ζ with $R_1(\infty) = R(z_0) = \infty$, so (as in the previous subsection) $R_1(\zeta) = P(\zeta) + H(\zeta)$ for some polynomial P with $H(\infty) = 0$. Unlike the singular part G_∞ of R itself, the constant term $P(0)$ here belongs to the *regular* behavior of R at z_0, not to its principal part (a Laurent principal part at a *finite* point never has a constant term) – so we discard it:

$$G_{z_0}(\zeta) := P(\zeta) - P(0), \qquad H_{z_0}(\zeta) := R_1(\zeta) - G_{z_0}(\zeta) = H(\zeta) + P(0),$$

so that $G_{z_0}(0) = 0$ and $H_{z_0}(\infty) = P(0)$.

Changing back via $\zeta = 1/(z - z_0)$, i.e. $R_1\big(1/(z - z_0)\big) = R(z)$, we get

$$R(z) - G_{z_0}\Big(\frac{1}{z - z_0}\Big) = H_{z_0}\Big(\frac{1}{z - z_0}\Big),$$

which is *regular* (finite, not a pole) at z_0 – it tends to $H_{z_0}(\infty) = P(0)$ as $z \to z_0$. The term $G_{z_0}\big(1/(z - z_0)\big)$ is a polynomial in $1/(z - z_0)$ with *no constant term*: among finite points it only has a pole at z_0, it vanishes as $z \to \infty$ (crucial for the assembly below), and $\deg G_{z_0}$ is exactly the order of z_0 as a pole of R.

Assembling the Decomposition

Let $z_1, \dots, z_h$ be all the finite poles of R, and set

$$\Psi(z) := R(z) - G_\infty(z) - \sum_{k=1}^{h} G_{z_k}\Big(\frac{1}{z - z_k}\Big).$$

Each $G_{z_k}\big(1/(z - z_k)\big)$ is a polynomial in $1/(z - z_k)$, so (by the previous subsection) it cancels the pole of R at z_k without introducing a pole anywhere else; combined with G_∞ cancelling the behavior at ∞, Ψ has no poles anywhere on the Riemann sphere. In particular Ψ has no finite poles, so it is a polynomial; and having no pole at ∞ either forces (by the classification of behavior at infinity discussed in the companion note) that polynomial to have degree 0 – i.e. $\Psi \equiv c$ for some constant c, with no need to invoke boundedness of a general entire function. To find c, let $z \to \infty$: $R(z) - G_\infty(z) \to 0$ by construction, and *each* $G_{z_k}\big(1/(z - z_k)\big) \to G_{z_k}(0) = 0$, since G_{z_k} was built with no constant term. Hence $\Psi(z) \to 0$, so $c = 0$. Therefore:

Partial Fraction Decomposition

$$R(z) = G_\infty(z) + \sum_{k=1}^{h} G_{z_k}\Big(\frac{1}{z - z_k}\Big),$$

where G_∞ is the singular part of R at ∞ and each G_{z_k} is the singular part of R at the pole z_k.